Clique-inverse graphs of bipartite graphs\footnote{This work was partially supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - FAPERJ, Brazil.}

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Abstract

The clique graph $K(G)$ of a given graph $G$ is the intersection graph of the collection of maximal cliques of $G$. Given a family $\mathcal{F}$ of graphs, the clique-inverse graphs of $\mathcal{F}$ are the graphs whose clique graphs belong to $\mathcal{F}$. In this work, we describe characterizations for clique-inverse graphs of bipartite graphs, chordal bipartite graphs, and trees. The characterizations lead to polynomial time algorithms for the corresponding recognition problems.

Keywords: intersection graphs, clique graphs, clique-inverse graphs

1 Introduction

Let $G$ be a finite undirected graph with no loops nor multiple edges. Denote the vertex set of $G$ by $V(G)$, and the edge set by $E(G)$. A subgraph $H$ of $G$ is a graph where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set $X$ of vertices of $G$, denote by $G[X]$ the subgraph of $G$
induced by $X$, that is, the vertex set of $G[X]$ is $X$ and two vertices are adjacent in it if they are so in $G$.

A clique is a subset of vertices inducing a complete subgraph of $G$, while a maximal clique is one not properly contained in any other. The clique number $\omega(G)$ of $G$ is the largest order of a clique in $G$.

A chord $c$ is an edge linking two non-consecutive vertices in a cycle. Denote by $C_k$ a cycle with $k$ vertices. A graph is chordal if it contains no induced subgraph isomorphic to $C_k$ for $k \geq 4$.

A graph is bipartite if its vertex set can be partitioned into two sets $U$ and $W$ such that every edge in $E(G)$ links a vertex of $U$ to a vertex of $W$. A graph is chordal bipartite if it is bipartite and contains no induced subgraph isomorphic to $C_{2k}$ for $k \geq 3$.

The clique graph $K(G)$ of $G$ is the intersection graph of the collection of maximal cliques of $G$. If $H = K(G)$, we say that $G$ is a clique-inverse graph of $H$. Given a family $\mathcal{F}$ of graphs, the family of clique-inverse graphs of $\mathcal{F}$ is defined as

$$K^{-1}(\mathcal{F}) = \{G|K(G) \in \mathcal{F}\}.$$ 

In [6], Hedetniemi and Slater presented characterizations for clique graphs of triangle-free graphs, bipartite graphs, and trees:

**Theorem 1** [6] Let $G$ be a graph. Then $G \in K(\mathcal{F})$ if and only if $K(G) \in \mathcal{F}$ and any two distinct maximal cliques of $G$ have at most one vertex in common, where $\mathcal{F}$ is one of the following families: triangle-free graphs, bipartite graphs, or trees. $\blacksquare$

Let $\mathcal{I}_k$ be the family of graphs with the following property: any two distinct maximal cliques of a graph in $\mathcal{I}_k$ have at most $k$ vertices in common. Then Hedetniemi and Slater's result can be rewritten as

$$K(\mathcal{F}) = K^{-1}(\mathcal{F}) \cap \mathcal{I}_1,$$

where $\mathcal{F}$ is one of the families cited in the above theorem. Although the problem of characterizing clique graphs of certain families has been studied for several cases, e.g. [1, 2, 5, 6, 7, 11, 15], much less is known about the corresponding inverse problem, which can be stated as follows: given a family $\mathcal{F}$ of graphs, characterize $K^{-1}(\mathcal{F})$, called
the family of clique-inverse graphs of $\mathcal{F}$. In this work, characterizations are described for clique-inverse graphs of bipartite graphs, chordal bipartite graphs, and trees. The characterizations lead to polynomial time algorithms for solving the corresponding recognition problems.

Clique-inverse graphs were the subjects of [9] and [12]. They are also called roots (relative to the clique operator), see e.g. [10]. Clique-inverse graphs of complete graphs are called clique-complete. A characterization of the minimal clique-complete graphs with no universal vertex (a vertex adjacent to all other vertices of the graph) has been formulated in [9]. It corresponds to a description of the minimal clique-complete graphs whose maximal cliques do not satisfy the Helly property. In [13], characterizations for clique-inverse graphs of triangle-free graphs and $K_4$-free graphs are presented in terms of forbidden subgraphs.

The following result is a characterization for clique-inverse graphs of triangle-free graphs. It will be used later:

**Theorem 2** [13] $G$ is a clique-inverse graph of a triangle-free graph if and only if $G$ does not contain as an induced subgraph any of the following graphs: $K_{1,3}$, 4-fan, 4-wheel (see Figure 1).

![Forbidden subgraphs](image)

Figure 1: Forbidden subgraphs for clique-inverse graphs of triangle-free graphs.

## 2 The characterizations

In this section we give complete characterizations for the situations in which $K(G)$ is bipartite, chordal bipartite, or a tree. We begin by
analyzing the case in which $K(G)$ is bipartite. The characterization
will be formulated in terms of a list of forbidden subgraphs.

Any bipartite graph is triangle-free. Thus, $K^{-1}(BIPARTITE)$ is
contained in $K^{-1}(\text{TRIANGLE} - \text{FREE})$.

**Theorem 3** A graph $G$ is a clique-inverse graph of a bipartite graph
if and only if $G$ does not contain as an induced subgraph any of the
following: $K_{1,3}$, 4-fan, 4-wheel, and $C_{2k+5}$ (for all $k \geq 0$).

**Proof.** ($\Rightarrow$): Assume by contradiction that $G$ is a clique-inverse
graph of a bipartite graph and $G$ contains $S = C_{2k+5}$ as an induced
subgraph. Write $S = u_0u_1 \ldots u_pu_0$, where $p = 2k+4$, $k \geq 0$. Clearly,
there exists a collection $\mathcal{M} = \{M_0, M_1, \ldots, M_p\}$ of maximal cliques
of $G$ such that each edge $e_i = \{u_i, u_{i+1}\}$ of $S$ lies in exactly one of
the cliques in $\mathcal{M}$, say $e_i$ lies in $M_i$ (indices are taken circularly in the
range $0 \ldots p$). Note that $u_{i+1} \in M_i \cap M_{i+1}$, that is, $M_i$ and $M_{i+1}$
intersect. Thus, $M_0M_1 \ldots M_pM_0$ is an odd cycle in $K(G)$. This is a
contradiction, since $K(G)$ is bipartite. On the other hand, by Theo-
rem 2, if $G$ contains either a 4-wheel, a 4-fan, or $K_{1,3}$ as an induced
subgraph, then $K(G)$ contains a triangle, another contradiction.

($\Leftarrow$): Assume by contradiction that $G$ does not contain any of the
graphs listed in the statement of the theorem as an induced subgraph,
and that $G$ is not a clique-inverse graph of a bipartite graph. Then,
there exists a chordless odd cycle $C = M_0M_1 \ldots M_{2p}M_0$ in $K(G)$,
where $p \geq 1$ and each $M_i$ is a distinct maximal clique of $G$. Choose
$C$ for which $p$ is minimum. There are two possible cases:

Case 1: $p = 1$. Then, $K(G)$ contains a triangle. This implies, by
Theorem 2, that $G$ contains either a 4-wheel, a 4-fan, or $K_{1,3}$ as an
induced subgraph, a contradiction.

Case 2: $p > 1$. This situation is depicted in Figure 2 (for $p = 2$).
Let $u_i \in M_i \cap M_{i+1}$, where indices are taken circularly in the range
$0 \ldots 2p$. Note that each $u_i$ belongs to no maximal cliques of $G$ other
than $M_i$ and $M_{i+1}$. Otherwise, if $u_i$ also belongs to a maximal clique
$M$ distinct from $M_i$ and $M_{i+1}$, then $K(G)$ contains a triangle, a
contradiction - since the cycle $C = M_0M_1 \ldots M_{2p}M_0$ in $K(G)$ has been
taken for $p > 1$ minimum. Thus, the cycle $C_G = u_0u_2 \ldots u_{2p}u_0$ in
$G$ is chordless, since the existence of a chord linking non-consecutive
vertices $u_k$ and $u_j$ in $C_G$ would imply the existence of a new maximal
clique $M$ containing $u_k$ and $u_j$, distinct from the cliques in the
multiset \( \{M_k, M_{k+1}, M_j, M_{j+1}\} \). Thus, \( C_G \) is a chordless odd cycle with \( 2p + 1 \geq 5 \) vertices, which contradicts the assumption that \( G \) does not contain \( C_{2k+5}, k \geq 0 \), as an induced subgraph. □

![Diagram](image)

Figure 2: Case 2 of Theorem 3, for \( p = 2 \).

In order to characterize clique-inverse graphs of chordal bipartite graphs, we employ an additional definition. Let \( C = v_0v_1\ldots v_kv_0 \) \((k \geq 3)\) be a cycle in a graph \( G \). We say that \( C \) admits an even division if there exists a vertex \( w \in G \setminus C \) which is adjacent to four distinct vertices \( v_i, v_{i+1}, v_j, v_{j+1} \) of \( C \) such that \( j - (i + 1) \) is even, that is, the path \( v_{i+1}v_{i+2}\ldots v_j \) has an even number of edges. The indices are taken circularly in the range \( 0 \ldots k \). See Figure 3.

In the cycle \( v_1v_2v_3v_4v_5v_6v_1 \) admits an even division.
The next theorem characterizes $K^{-1}(\text{CHORDAL BIPARTITE})$ in terms of $K^{-1}(\text{BIPARTITE})$:

**Theorem 4** A graph $G$ is a clique-inverse graph of a chordal bipartite graph if and only if $G \in K^{-1}(\text{BIPARTITE})$ and every chordless even cycle of $G$ with at least six vertices admits an even division.

**Proof.** ($\Rightarrow$): Assume that $G$ is a clique-inverse graph of a chordal bipartite graph, that is, $K(G)$ is chordal bipartite. Clearly, $G \in K^{-1}(\text{BIPARTITE})$. Now, let $C = v_0v_1 \ldots v_{2k-1}v_0$ be a chordless even cycle of $G$ with $k \geq 3$. Let $M_i$ be a maximal clique of $G$ containing the edge $\{v_i, v_{i+1}\}$, where indices are taken circularly in the range $1 \ldots 2k-1$. It is clear that $M_0M_1 \ldots M_{2k-1}M_0$ is a cycle in $K(G)$, since $v_i \in M_{i-1} \cap M_i$. Assume by contradiction that $C$ does not admit an even division. Then $M_i \cap M_j = \emptyset$ for non-consecutive indices $i$ and $j$ such that $j - (i+1)$ is even. Observe that $M_i \cap M_j = \emptyset$ also holds for non-consecutive $i$ and $j$ such that $j - (i+1)$ is odd, since otherwise $K(G)$ would contain an odd cycle, contradicting $K(G)$ to be bipartite. It follows that $M_0M_1 \ldots M_{2k-1}M_0$ is chordless, $k \geq 3$. This contradicts the fact that $K(G)$ is chordal bipartite.

($\Leftarrow$): Assume that $G \in K^{-1}(\text{BIPARTITE})$ and every chordless even cycle of $G$ with at least six vertices admits an even division. Then, $K(G)$ is bipartite. Now, let us show that $K(G)$ does not contain an induced subgraph isomorphic to $C_{2k}$, for $k \geq 3$. Assume by contradiction that $C = M_0M_1 \ldots M_{2k-1}M_0$ is a chordless cycle in $K(G)$ for $k \geq 3$. Then, there exists a cycle $C_G = v_0v_1 \ldots v_{2k-1}v_0$ in $G$ such that the edge $\{v_i, v_{i+1}\}$ lies in the maximal clique $M_i$, $0 \leq i \leq 2k-1$, where $i$ is taken circularly in the range $0 \ldots 2k-1$. By the assumption, $C_G$ admits an even division. Therefore, let $w \in G \setminus C$ adjacent to four distinct vertices $v_i, v_{i+1}, v_j, v_{j+1}$ of $C$ such that $j - (i+1)$ is even. Observe that $v_i$ belongs to no maximal cliques other than $M_{i-1}$ and $M_i$, for otherwise $K(G)$ would contain a triangle. Since $w, v_i,$ and $v_{i+1}$ belong to a same maximal clique, it follows that $w$ belongs to at least one of the cliques $M_{i-1}$ and $M_i$. Analogously, $w$ belongs to at least one of the cliques $M_{j-1}$ and $M_j$. Thus, some clique of the set $\{M_{i-1}, M_i\}$ intersects at least one clique of the set $\{M_{j-1}, M_j\}$. Since $j - (i + 1) > 0$, it follows that there exist two intersecting maximal cliques with non-consecutive indices in the cycle.
\( C = M_0 M_1 \ldots M_{2k-1} M_0 \). This is a contradiction, since \( C \) has been assumed to be chordless.

To conclude this section, let us examine the family \( K^{-1}(TREE) \). The following definition will be employed: a **domino** is a graph where every vertex belongs to at most two distinct maximal cliques \([8]\).

**Theorem 5** A graph \( G \) is a clique-inverse graph of a tree if and only if \( G \) is a chordal domino.

**Proof.** (\( \Rightarrow \)): Assume by contradiction that \( G \) is a clique-inverse graph of a tree and \( G \) is not chordal, and let \( v_0 v_1 \ldots v_k v_0 \) be a chordless cycle in \( G \) with \( k \geq 3 \). Then there exist \( k \) distinct maximal cliques \( M_0, M_1, \ldots, M_k \) in \( G \) such that the edge \( \{v_i, v_{i+1}\} \) lies in \( M_i \) and \( M_i \) intersects \( M_{i+1} \), where the indices are taken circularly in the range \( 0 \ldots k \). This implies that \( M_0 M_1 \ldots M_k M_0 \) is a cycle in \( K(G) \). But this is a contradiction, since \( K(G) \) is assumed to be a tree.

(\( \Leftarrow \)): Assume by contradiction that \( G \) is a chordal domino and \( K(G) \) is not a tree. Let \( M_0 M_1 \ldots M_k M_0, k \geq 2 \), be a cycle in \( K(G) \). There are two possible cases.

**Case 1:** \( k = 2 \).

Let \( R = M_0 \cap M_1 \cap M_2 \). It is clear that \( R = \emptyset \), since every vertex of \( G \) belongs to at most two maximal cliques. Let \( v_{01} \in M_0 \cap M_1 \), \( v_{02} \in M_0 \cap M_2 \), and \( v_{12} \in M_1 \cap M_2 \). Observe that \( v_{01}, v_{02}, \) and \( v_{12} \) induce a triangle in \( G \). Therefore, there exists a maximal clique \( M \) in \( G \) containing \( v_{01}, v_{02}, \) and \( v_{12} \). Clearly, \( M \neq M_0 \), since \( v_{12} \in M \) and \( v_{12} \notin M_0 \). Analogously, \( M \neq M_1 \). This implies that \( v_{01} \) belongs to \( M_0, M_1 \), and \( M \), contradicting the fact that every vertex of \( G \) belongs to at most two maximal cliques.

**Case 2:** \( k > 2 \).

Let \( v_i \in M_i \cap M_{i+1} \), where \( 0 \leq i \leq k \) and indices taken circularly in the range \( 0 \ldots k \). Then, \( C = v_0 v_1 \ldots v_k v_0 \) is a cycle in \( G \). The \( v_i \)’s are distinct, for otherwise, if \( v_i = v_j \) for \( i \neq j \), then \( v_i \) would belong to
all the cliques in the multiset \( \{M_i, M_{i+1}, M_j, M_{j+1}\} \), which contains at least three distinct elements. Since \( G \) is chordal, \( C \) has a chord joining \( v_r \) and \( v_{r+2 \mod k} \), for some \( r \) in the range \( 0 \ldots k \). Thus, \( v_r \), \( v_{r+1} \), and \( v_{r+2} \) induce a triangle in \( G \). This implies that there exists a maximal clique \( M \) in \( G \) containing these three vertices. Clearly, \( M \neq M_r \), since \( v_{r+1} \in M \) and \( v_{r+1} \notin M_r \). Analogously, \( M \neq M_{r+1} \). Thus, \( v_r \) belongs to \( M_r \), \( M_{r+1} \), and \( M \). This contradicts the fact that every vertex of \( G \) belongs to at most two distinct maximal cliques. 

\( \square \)

**Corollary 6** Let \( G \) be a graph. Then, \( G \) is a clique-inverse graph of a tree if and only if \( G \) does not contain as an induced subgraph any of the following graphs: \( K_{1,3} \), 4-fan, 4-wheel, \( C_k \) (for all \( k \geq 4 \)).

**Proof.** If \( G \in K^{-1}(\text{TREE}) \), then \( G \) is a clique-inverse graph of a triangle-free graph. Therefore, by Theorem 2, \( G \) does not contain as an induced subgraph any of the following graphs: \( K_{1,3} \), 4-fan, 4-wheel. Moreover, by Theorem 5, \( G \) is chordal, and thus the first part follows. Conversely, if \( G \) does not contain \( K_{1,3} \), 4-fan, 4-wheel, or \( C_k \) (\( k \geq 4 \)) as an induced subgraph, then \( G \) is chordal. Moreover, by Theorem 2, \( K(G) \) contains no triangle, which implies that each vertex of \( G \) can belong to at most two distinct maximal cliques, that is, \( G \) is a domino. Thus, by Theorem 5, \( G \) is a clique-inverse graph of a tree. \( \square \)

3 Algorithms

We start this section by observing that if \( K(G) \) has bounded clique number, then \( |V(K(G))| \) is \( O(n) \), that is, the number of maximal cliques of \( G \) is linearly bounded.

**Lemma 7** [14] Let \( G \) be a connected graph. If \( \omega(K(G)) \leq r \) for a positive constant \( r \), then \( |V(K(G))| \leq rn \).

**Proof.** Observe that any vertex \( v \) of \( G \) may belong to at most \( r \) maximal cliques, since otherwise the cliques of \( G \) containing \( v \) would correspond to a clique of size at least \( r+1 \) in \( K(G) \), a contradiction.
Therefore, the number of maximal cliques of $G$ is at most $rn$, that is, $|V(K(G))| \leq rn$. 

A consequence of the above lemma is the fact that clique-inverse graphs of bipartite graphs have few maximal cliques.

**Corollary 8** Let $G$ be a connected graph. If $K(G)$ is bipartite then $G$ contains at most $2n$ maximal cliques. 

By using the above observations, we describe below polynomial-time recognition algorithms for the families focused in this work.

Let $G$ be a graph. In order to decide whether or not $K(G)$ is bipartite, first check whether $G$ contains at most $2n$ maximal cliques by applying the algorithm in [16] to \overline{G}, which generates all the maximal cliques of $G$ with delay $O(nm)$, where $m = |E(G)|$. This task takes $O(n^2m)$ time. If $G$ has more than $2n$ maximal cliques, then the answer to the question 'Is $G$ in $K^{-1}(BIPARTITE)$?' is clearly 'no'. Otherwise, construct $K(G)$ by taking the maximal cliques generated by the algorithm. This task takes $O(nm)$ time, since $G$ has at most $2n$ maximal cliques, and each intersection test between two cliques takes $O(m)$ time. Finally, verify whether $K(G)$ is bipartite in $O(m)$ time (recall that $|V(K(G))|$ is $O(n)$). Therefore, the entire procedure answers the question 'Is $G$ in $K^{-1}(BIPARTITE)$?' in polynomial time.

In order to decide whether $K(G)$ is chordal bipartite, apply a similar algorithm. The graph $K(G)$ can be constructed in polynomial time. In addition, $K(G)$ can be recognized as a chordal bipartite graph also in polynomial time [4].

Finally, one may use the characterization of Theorem 5 to verify whether $K(G)$ is a tree. Checking chordality and recognizing whether $G$ is a domino can be done in polynomial time.

**References**


